An Investigation into Some Aspects of Braess' Paradox

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October 2006

1 Introduction

The most common form of assignment is the equilibrium assignment which is based on Wardrop's first principle of route choice which says the following [16]:

The journey times in all routes used are equal and less than those which would be experienced by a single vehicle on any unused route.

In 1968 Braess [4], [5] presented a network, where, when an extra link was added to the network, resulted in the apparent paradox that the total travel time increased on the network.

This paper starts by presenting Braess' example. Two other examples of apparent paradoxes are also presented. The paper also includes some references to Braess' paradox from the literature and compares them with results that were obtained using both a real-world network and a small one that was constructed by the author.

The paper concludes with an example where Braess' paradox was eliminated.

2 Braess' Paradox

In 1968 Braess [4], [5] presented an example of an equilibrium assignment problem that produces an apparently paradoxical result. In this example, the addition of a new link to the network results in an increase in the total travel time on the network, instead of the decrease that would be intuitively expected. This phenomenon has become known as Braess' Paradox and is discussed below.

Figure 1 shows a simple network including one O-D pair that is connected by four links and two paths. The figure shows the two paths (numbered 1 and 2) and the link performance functions for the four links. Assume that six units of flow travel between O and D (ie q=6). The user equilibrium flow pattern for this network can be solved by inspection (due to the travel time symmetry of the two paths). It is obvious that half of the flow would use each path and that the solution would be:

$$f_1 = f_2 = 3 \text{ flow units}$$

or, in terms of link flows,

$$x_1 = x_2 = x_3 = x_4 = 3 \text{ flow units.}$$

The associated link travel times are:

$$t_1 = 53, t_2 = 53, t_3 = 30, t_4 = 30 \text{ time units}$$

and the path times are

$$c_1 = c_2 = 83 \text{ time units, satisfying the user equilibrium criterion.}$$

The total travel time on the network is 498 (flow-time) units.

Figure 2 shows the network expanded to include a new link connecting the two intermediate nodes. The figure shows this added (fifth) link, the performance function for this link, and the new path (number 3) resulting from the addition of the link.

The old UE flow pattern is no longer an equilibrium solution since,

$$x_1 = 3, x_2 = 3, x_3 = 3, x_4 = 3, x_5 = 0 \text{ flow units}$$

with path travel times being

$$c_1 = 83, c_2 = 83, c_3 = 70 \text{ time units.}$$

The travel time on the unused path (path 3) is lower than the travel times on the two used paths so this is not an equilibrium solution. Figure 2 shows a possible sequence of assignment of flow units that would result in an equilibrium solution. The equilibrium flow pattern for the new network is given by the solution

$$x_1 = 2, x_2 = 2, x_3 = 2, x_4 = 2, x_5 = 2 \text{ flow units}$$

with path flows

$$f_1 = f_2 = f_3 = 2 \text{ flow units}$$

and path travel times

$$c_1 = c_2 = c_3 = 92 \text{ time units.}$$

It is important to note that the total travel time on the network in now 552 (flow-time) units compared to the 498 (flow-time) units before the fifth link was added. Therefore, the addition of the link has resulted in the travel time of each traveller increasing from 83 to 92 time units and

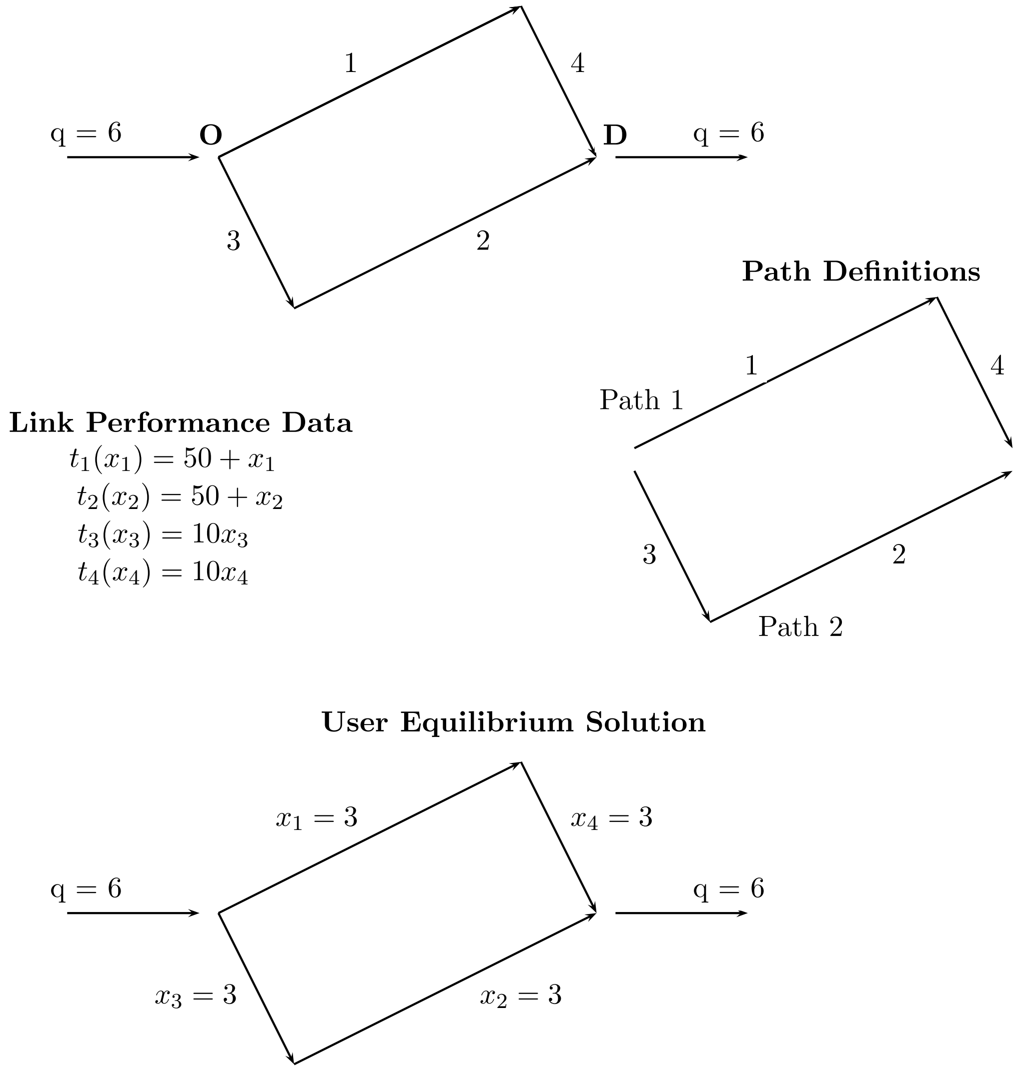


Figure 1: Intial Braess' Network with Equilibrium Flows

the total travel time increasing from 498 (flow time) units to 552 (flow time) units. The addition of the new link has therefore made the situation worse. This apparently counter intuitive result is known as Braess' Paradox.

Figures 1 and 2 show the three paths that traffic can follow in the augmented network. The above analysis was based on Sheffi [13].

There is some evidence that this situation may well occur in practice. Knödel [8] cited in Murchland [10] describes a case in Stuttgart. Major road investments in the city centre, in the vicinity of the Schlossplatz, failed to yield the benefits that had been expected. The benefits were only obtained when a cross street, the lower part of Königstrasse, was withdrawn from use by traffic.

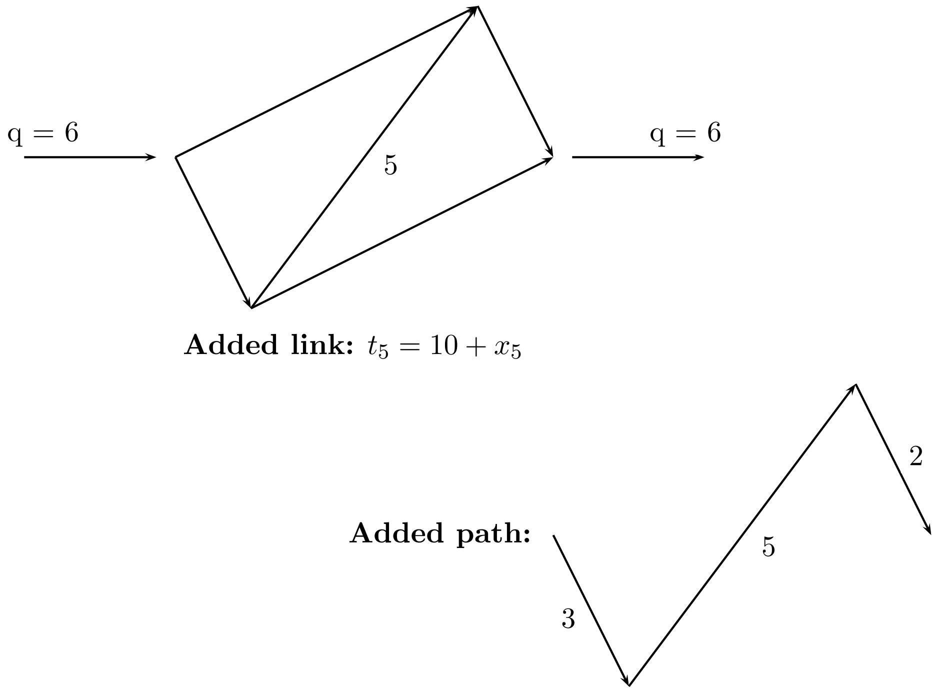


Figure 2: Braess' Network with Additional Link and New Solution

3 Other Paradoxes

Two other paradoxes which may be of interest to the reader are described below.

3.1 Downs-Thomson

The Downs-Thomson paradox [7], [15] refers to a situation where an origin-destination pair is served by two modes.

Suppose there is a roadway along a route which is served also by rail. If travel costs by road rise with the number of drivers, while rail costs are independent of ridership; then in equilibrium, just that number of travellers will drive to raise travel costs by road to equal costs by rail. I road capacity is now expanded, users will shift to the road until it as congested as before.

If the rail service has to balance its budget, the loss of revenue will force it to increase fares and/or cut service, inducing more passengers to switch to road. Since the rail service is now worse than it was, the system will only be in equilibrium when the road is more congested than it was previously.

Therefore an improvement to the road results in the users of both modes being worse off than they were initially.

3.2 Mechanical Analogue of Braess Paradox

Cohen and Horowitz [6] proposed what they termed a mechanical analogue of Braess' paradox. This is shown in Figure 3. In part (a) of Figure 3 the system consists of two springs in series, P-A and B-Q, that are connected by a string between A and B. There are also two "safety" strings between P and B and between A and Q. A weight with a mass of M is suspended from Q.

The question is what happens to the position of the weight with mass, M, if the string A-B is cut? Cohen and Horowitz showed that for certain combinations of strength of springs, length of strings and mass of weight, the weight will rise instead of dropping as could be expected.

This situation can be explained by the fact that in the case shown in part (a) of the figure the springs are in series and each carry the full mass M. In part(b) of the figure, the springs are in parallel and support only half of the mass M.

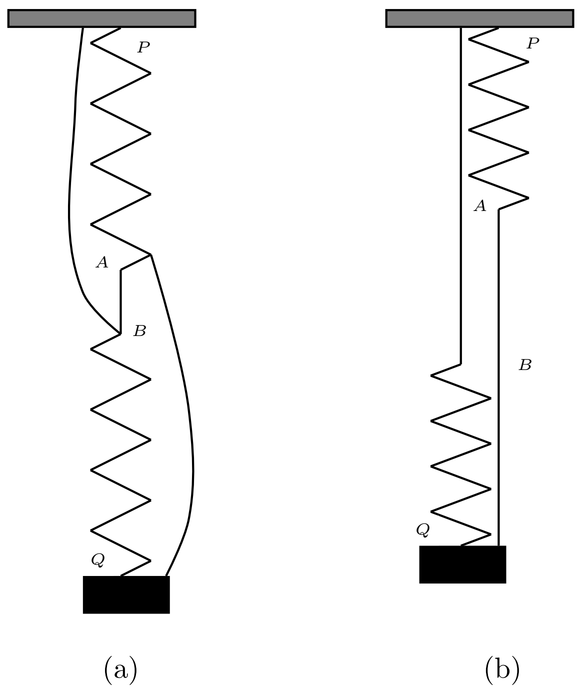


Figure 3: Mechanical Analogue of Braess' Paradox

4 How Prevelant is Braess' Paradox

In real world networks, it is generally not possible to predict whether adding a new link will results in Braess' paradox or not. LeBlanc [9] said the following in this regard: "When dealing with a network with many origins and destinations, it is not clear whether adding an arc will increase or decrease the congestion at equilibrium."

Steinberg and Zangwill [14] produced a theorem showing that, under what they term "reasonable assumptions", Braess' paradox is as likely to occur as not. Whether this is always the case in real world networks was tested and the results are described below.

In order to test the prevalence of Braess' paradox, tests were done using data from the 1985 PWV Update Study [12]. In this study which had 1985 as the base year, a network was developed for the year 2000. This was done by adding road improvements for each year from 1988 to 2000. In the tests, the results of which follow, each year was tested separately by assigning a matrix which was derived by interpolating between the matrices which had been developed as part of the study for 1985 and 2000.

For each year the road improvements for that year as well as for the preceding years were tested to see whether they resulted in Braess' paradox or not. For example, there were 23 road projects for 1988 and another eight were added for 1989, all 31 projects were then tested for 1989. The reason for doing this was to check to see whether any of the projects for 1988 showed Braess' paradox in 1989 or later years.

Table 1 shows the results that were obtained using the default stopping criteria in EMME/2 for selected years. The "Largest Braess' Paradox" column contains the increase in the number of vehicle-hours resulting from including the project. In this case the applicable criterion was the number of iterations which was 15. However, Bloy [2] recommended that when the results of an equilibrium assignment are to be used in financial analyses, a stopping criterion of relative gap = 0.01 should be used. Similar recommendations were made by Blaschuk and Hunt [1], and Boyce, et al [3].

Table 1: Braess' Paradox in 1985 Update Study with 15 Iterations

Year	**No. of Projects**	**No. of Braess' Projects**	**Largest Braess' Paradox**
1989	31	1	13
1992	64	28	62
1993	7	19	58
1995	107	9	20
1996	123	71	70
1998	158	52	65
2000	186	79	84

The results obtained when using a relative gap of 0.01 as the stopping criterion are shown in Table 2. When comparing the results shown in Tables 1 and 2, it can be seen that in the majority of cases the number of projects showing Braess' paradox is greatly reduced, as is the size of the effects of the paradox.

Table 3 shows how having a more stringent stopping criterion results in a reduction in both the number of projects showing Braess' paradox and the size of the paradox (increase in vehicle-hours due to including the project).

Table 2: Braess' Paradox in 1985 Update Study with Relative Gap = 0.01

Year	**No. of Projects**	**No. of Braess' Projects**	**Largest Braess' Paradox**
1989	31	2	6
1992	64	2	1
1993	7	5	40
1995	107	0	-
1996	123	1	1
1998	158	1	1
2000	186	3	1

Table 3: Influence of Stopping Criterion on Braess' Paradox: 1996 - 123 projects

Stopping Criterion	**No. of Braess' Projects**	**Largest Braess' Paradox**
15 Iterations	71	70
Rel Gap = 0.20	29	16
Rel Gap = 0.10	0	-
Rel Gap = 0.05	4	5
Rel Gap = 0.03	2	4
Rel Gap = 0.01	1	1

5 Range of Values where Braess' Paradox Occurs

Pas and Pricipio [11] analysed the original Braess' network (see Figures 1 and 2) and determined that Braess' paradox occurs only in the range of $2.58 < Q < 8.89$ where Q is the total demand on the network.

They also produced the graphical representation of the flows on the three routes through the network shown in Figure 4. Routes 1 and 2 refer to the two routes in both the original and augmented networks, while route 3 included the added link.

It is interesting to note that there is no flow on the new link once the demand reaches the upper limit of the range where Braess' paradox occurs.

Figure 4 and the analysis of Pas and Principio raised the following questions:

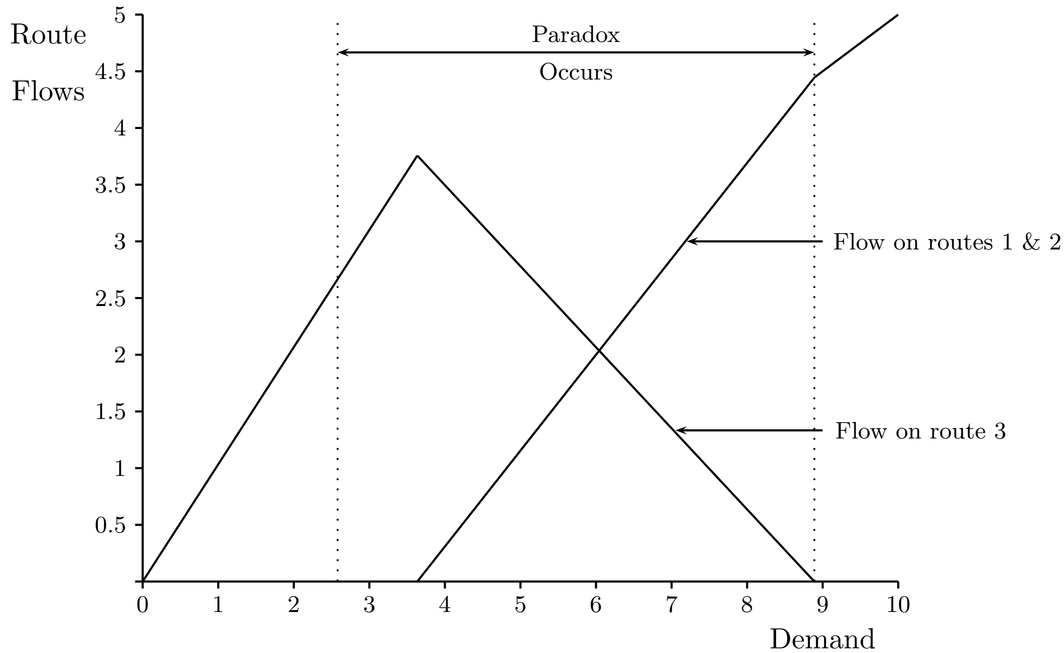


Figure 4: Route flows as a function of demand - Braess' example

- The functions in the Braess' network are all linear. What would the form of the plot showing the flows on the different routes be when using non-liner (e.g. BPR) delay functions?
- The Braess' functions contain no reference to the capacity of the links. What are the levels of congestion where Braess' paradox occurs?
- Does Braess' paradox occur only over a range of values in "real" networks?

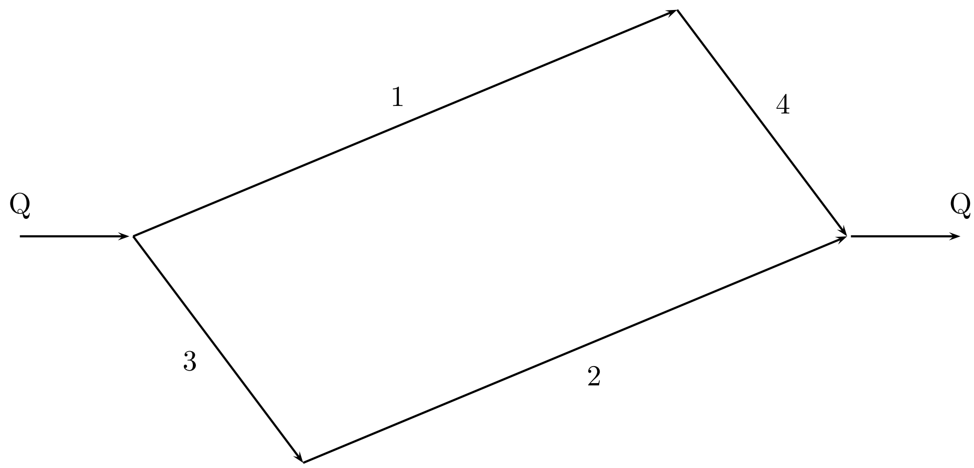
The first step in attempting to answer these questions was to construct a Braess' paradox type network using BPR delay functions. In order to obtain some idea as to whether it is necessary to have high levels of congestion, we have constructed a simple network that uses the link performance functions from a real world model. This network is shown in Figure 5 and the details of the links are provided in Table 4.

The Bureau of Public Roads (BPR) formula is used in many models and has the following form:

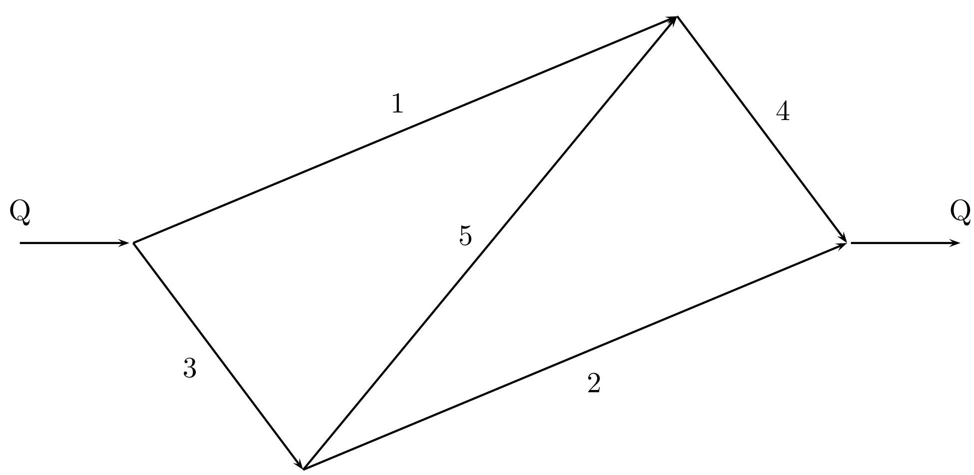
$$t = t_0[1 + a(x/c_p)^b]$$

where:

t	=	travel time under congested conditions
t_0	=	free flow travel time
x	=	assigned link volume
c_p	=	practical capacity, = 0.75 of the nominal capacity
a, b	=	constants, values of 0.15 and 4 were used



(a) Original network



(b) Augmented network

Figure 5: Braess type network with BPR link performance functions

This network was analysed using EMME/2 and the flows on the different routes through the network are shown in Figure 6.

It was found that Braess' paradox occurred only for the range $508.25 < Q < 873.99$. Unlike in the Braess network, there is flow on the additional link once the upper limit of the range is passed.

Table 4: Link characterstics

Link	Length (km)	Free flow speed (km/h)	Capacity (vph)
1	1.56	60	830
2	1.56	60	830
3	0.75	70	920
4	0.75	70	920
5	1.56	110	1110

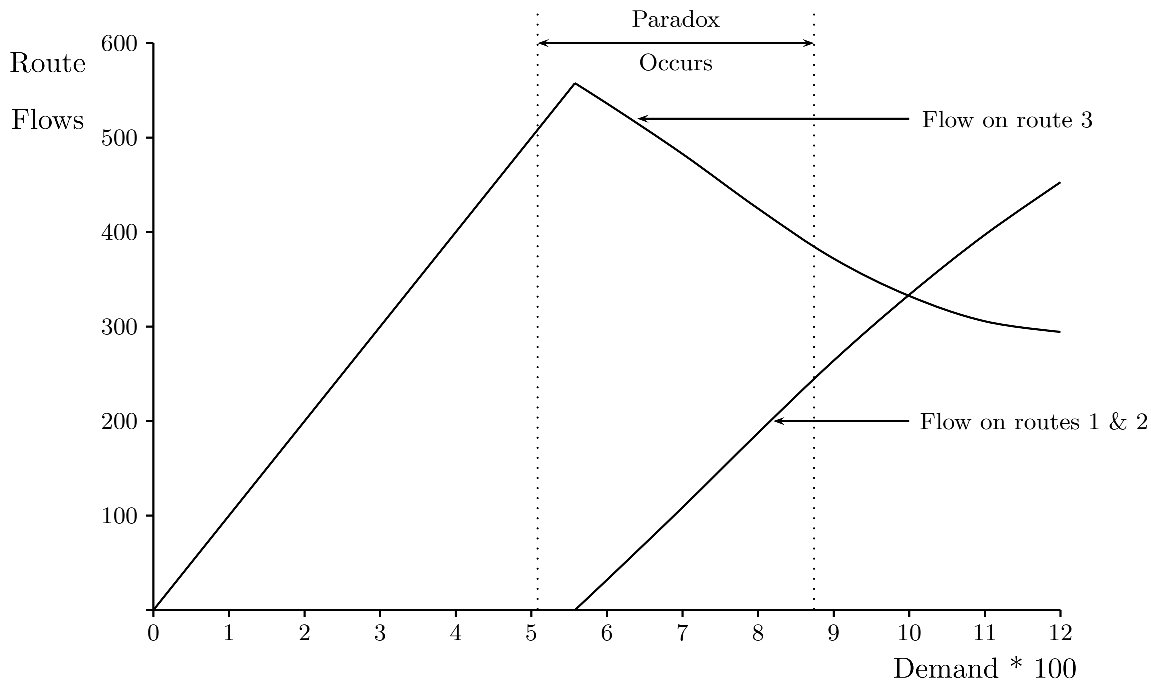


Figure 6: Route flows as a function of demand - BPR example

Having established the range of values over which Braess' paradox occurs in our BPR network, it was possible to determine the volume/capacity ratios (V/C) on the various links of the network at the limits of the range. These are shown in Table 5. The highest V/C ratio is 0.68. This is relatively low and would probably be considered to be a good operating condition.

Another way of representing the range of values for the total demand where Braess'paradox occurs is shown in Figure 7. In this figure the difference in costs on the original network and the augmented network is plotted against the total demand. If this difference is negative, i.e. the cost on the augmented network is higher than the cost on the original network, then Braess' paradox occurs.

An attempt was made to determine whether Braess' paradox occurs over a range of values in a real world network. Once again data from the 1985 PWV Update Study [12] was used. In the first attempt, the entire trip matrix was multiplied by a range of factors and the effect of

Table 5: Link flows and v/c ratios at which Braess' paradox occurs

Link	**Capacity (vph)**	**Flow where paradox starts**	**v/c**	**Flow where paradox ends**	**v/c**
1	830	0	0	245.15	0.30
2	830	0	0	245.15	0.30
3	920	508.25	0.55	628.85	0.68
4	920	508.25	0.55	628.85	0.68
5	1110	508.25	0.46	383.71	0.35

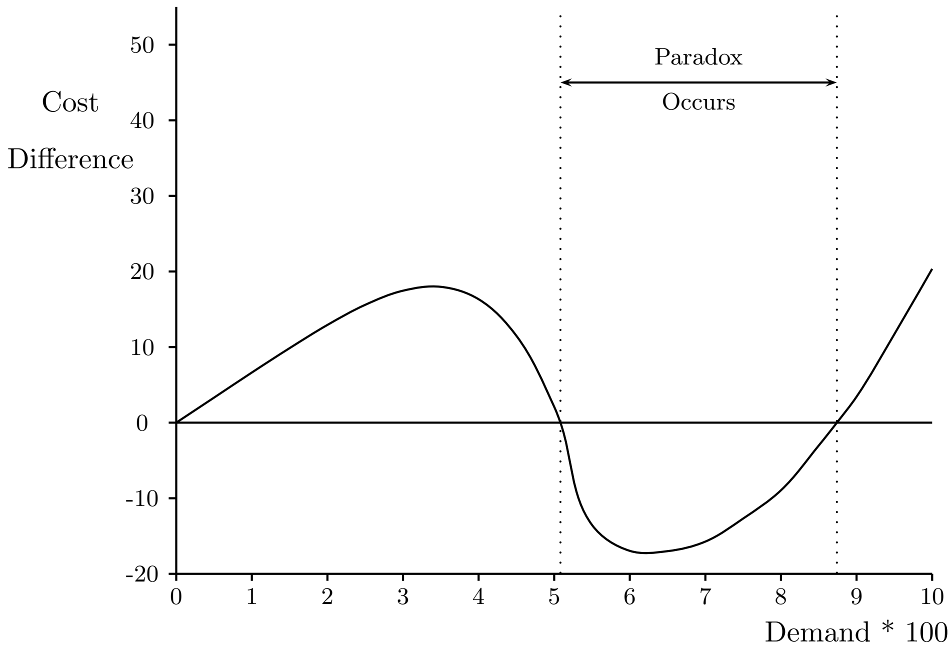


Figure 7: Cost on original network - cost on augmented network

removing projects from the network was tested using these matrices. The results of this exercise are shown in Table 6 for selected projects. In this table a negative value indicates the occurrence of Braess' paradox.

The first two columns (projects 57 and 26) appear to indicate the S-shaped curve shown in Figure 7 but without ever having negative values. Projects 18 and 21 start off showing Braess' paradox at low volumes but this soon disappears at higher volumes. Project 76 always show Braess' paradox that gets steadily worse.

A second test was carried out where at attempt was made to isolate the trips using the links in question. This was done by doing select link assignments on the specific projects and then applying factors to only the matrices obtained from the select links. These factored matrices were then combined with the remainder of the trip matrix and the projects were tested for Braess'

Table 6: Differences with and without projects when trip matrix is multiplied by a factor

Project	**57**	**26**	**18**	**21**	**28**	**76**
*0.5	3	10	-2	-1	9	-9
*0.6	8	24	-1	4	14	-12
*0.7	9	24	2	8	18	-16
*0.8	6	16	4	12	20	-21
*0.9	4	14	4	22	21	-28
*1.0	11	16	10	35	26	-40
*1.1	11	18	11	45	28	-41
*1.2	12	28	11	56	34	-54
*1.3	14	45	22	72	39	-57
*1.4	19	33	21	101	49	-80
*1.5	25	26	20	137	53	-93

paradox.

The results of this process are shown in Table 7 for selected projects. In this case it is not possible to see any indication of the S-shaped curve. It is however, possible to make the following observations:

- The occurrence of Braess' paradox disappears at higher volumes.
- Braess' paradox even disappears in the case of project 76 where it had got worse at higher volumes in the previous test.

Table 7: Differences with and without projects when select link matrix is multiplied by a factor

Project	**34**	**38**	**58**	**69**	**76**
*0.5	-1	0	-1	-1	-49
*0.7	0	0	0	0	-42
*0.9	-2	-1	-1	0	-41
*1.1	0	0	1	0	-39
*1.3	2	2	1	2	-23
*1.5	4	0	0	0	-23
*1.7	7	-1	-2	1	-30
*1.9	6	4	1	3	-26
*2.1	10	4	3	3	-15
*2.3	15	4	6	5	-4
*2.5	17	4	12	7	7

6 A Possible Methodology to Eliminate Braess Paradox

Figure 8 shows the three paths traffic can follow in the augmented network (paths 1 and 2 are also the paths that are used in the original network).

If one considers the paths shown in Figure 8, then it is obvious that, due to symmetry, the flows on paths 1 and 2 will be equal at equilibrium (the lengths and capacities of these links are the same).

In the case of the original network, the flows on all the links will be $0.5Q$.

In the case of the augmented network, if the flow on link 5 is P (i.e. on path 3), then the flows on the various links are as follows:

$$\text{Link1} = 0.5Q - 0.5P$$

$$\text{Link2} = 0.5Q - 0.5P$$

$$\text{Link3} = 0.5Q + 0.5P$$

$$\text{Link4} = 0.5Q + 0.5P$$

$$\text{Link5} = P$$

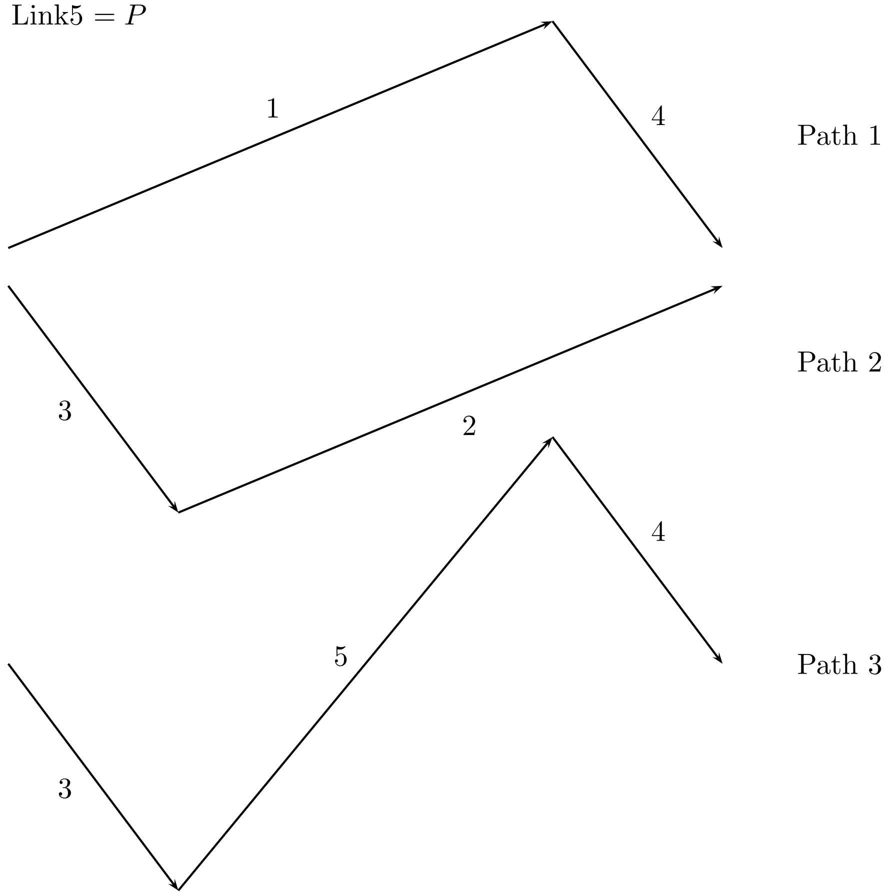


Figure 8: Paths through the augmented network shown in Figure 5

With the additional link in the augmented network, links 3 and 4 carry more flow than they would have in the original network (if there is any flow on path 3). Therefore, for Braess' paradox to occur the additional cost caused by the extra loading on links 3 and 4 exceeds the reduced costs by having lower loads on links 1 and 2 and the new link.

As a result of the above, increasing the capacity of links 3 and 4 should reduce the probability of Braess' paradox occurring. This was tested by increasing the capacity of links 3 and 4 in steps of ten per cent of their original capacities. This was found to be the case, with the range of values over which Braess' paradox decreasing until it no longer occurred. The results of this analysis are shown in Figure 9.

Figure 9 was derived using the EMME/2 computer program. However, it is also possible to derive the family of curves shown in Figure 9 analytically.

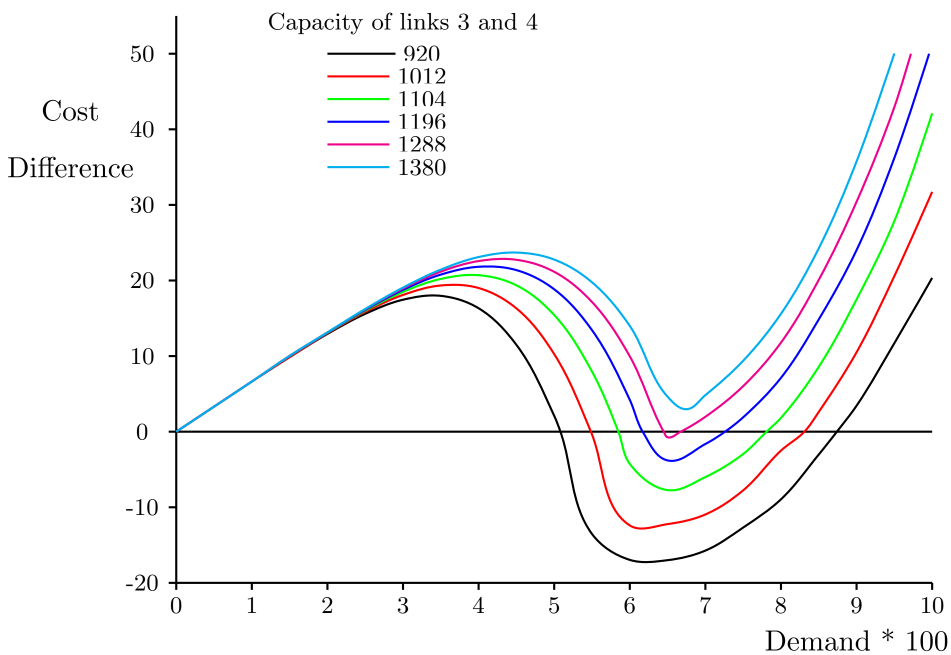


Figure 9: Cost on original network - cost on augmented network for different capacities on links 3 and 4

If it is possible to eliminate Braess' paradox from the small BPR network as described above, it may well be possible to do so in a real-world situation. Figure 10 shows a portion of the PWV Update network. In this case the road shown in blue was a two-lane, two-way road which was replaced by a four-lane divided road on a different alignment (shown in red). Braess' paradox occurred and including the new road resulted in an increase in the total travel time of 6 vehicle hours.

The volume-capacity ratios of the two networks were compared and the results are shown in Figure 11 where red indicates an increase in V/C and green shows a decrease. Increasing the

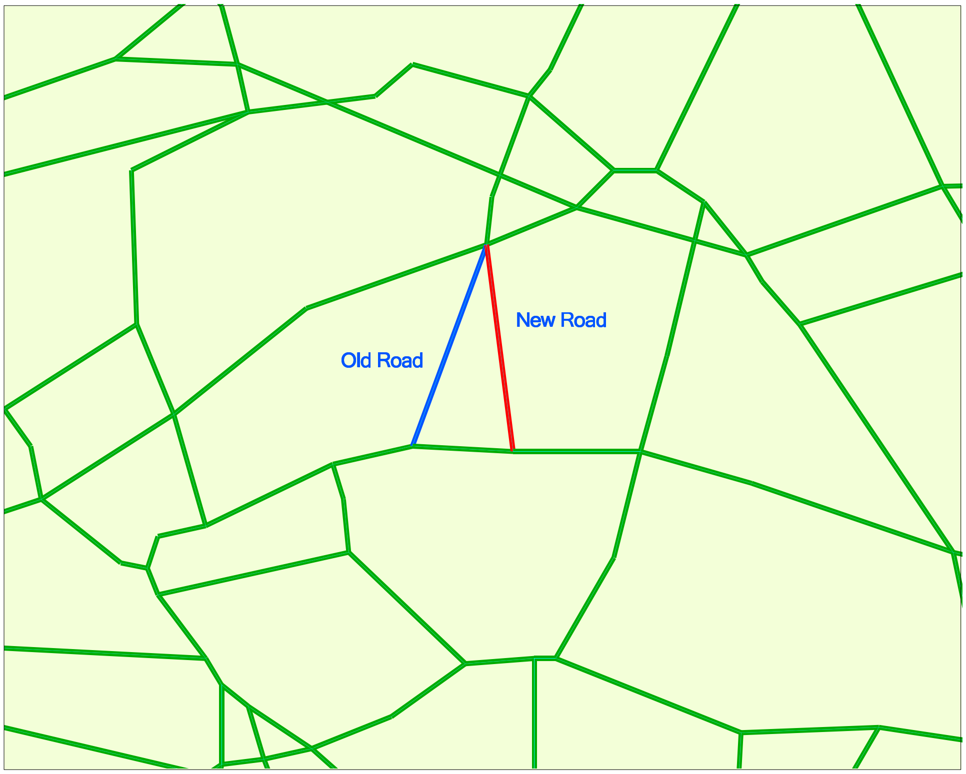


Figure 10: Change to Road Network

capacity by adding an extra lane on the short section of road to the left of the new road where the V/C ratio increased by 0.65 results in the elimination of Braess' paradox. The total travel time is now reduced by 3 vehicle-hours. Adding a lane to the section of road to the right of the new road reduces the travel time by a further 4 vehicle-hours. The sections where the lanes were added were on the schedule to be widened a couple of years later so Braess' paradox could be eliminated by including them in the network at an earlier date.

This methodology was tested on some other examples where it was also found that Braess' paradox could be eliminated.

7 Conclusions

The following conclusions can be drawn from the work described in this paper:

- Braess' paradox is less likely to occur if stringent stopping criteria such as a relative gap of 0.01 are used.
- Braess' paradox is less likely to occur at higher volumes.

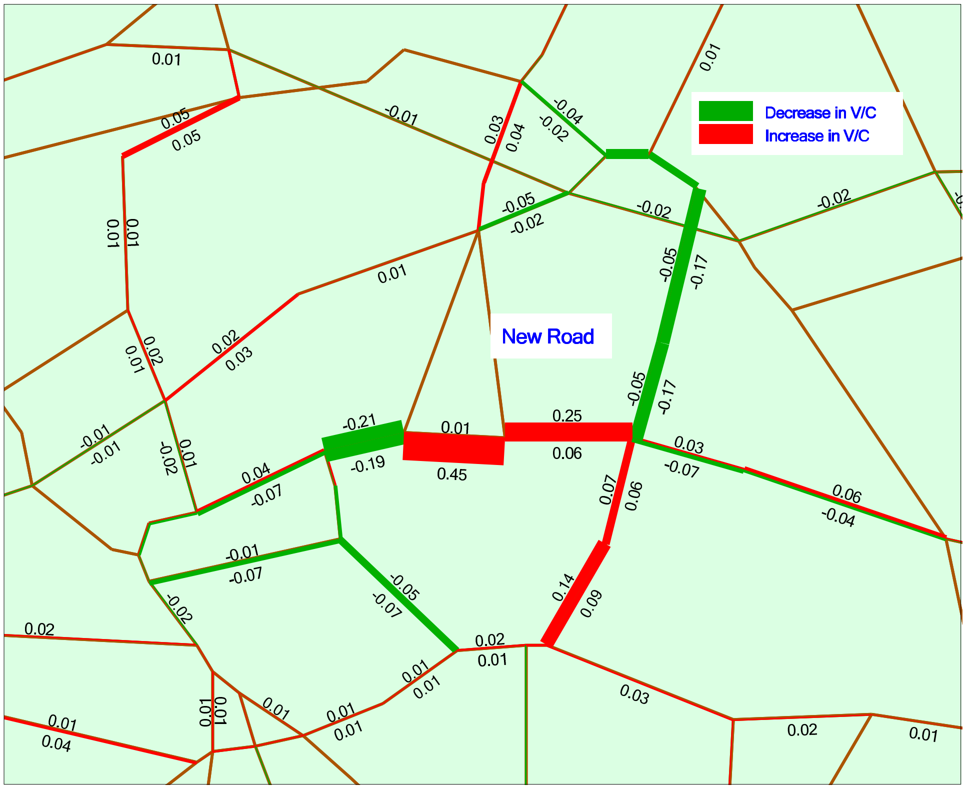


Figure 11: Changes in V/C Ratios

- It may be possible to eliminate Braess' paradox by comparing the volume capacity ratios and increasing the capacity of the roads where there is a large increase in the V/C ratio.

8 Acknowledgements

The author would like to acknowledge the following:

- The Gauteng Department of Public Transport, Roads and Works for permission to use their model data.
- the firm of Vela VKE Consulting Engineers for allowing me to use their EMME/2 licence.
- Professor P H Potgieter of Unisa for his suggestions and advice.

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